Tetrahedron equation
and cyclic quantum dilogarithm identities

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Abstract

We establish a hierarchy of quantum dilogarithm identities associated to a sequence of triangular shaped quivers. The tetrahedron equation plays a key role in our construction.

1 Introduction. Main results.

Fix $q \in (0,1)$. The *quantum exponential function* is the following formal series

$$\langle x \rangle_q = \sum_{n=0}^{\infty} \frac{(-x)^n}{(1-q)\dots(1-q^n)} \,. \tag{1}$$

It is well known that if X and Y are two q–commuting indeterminates, that is they satisfy the commutation relation

$$\mathsf{Y}\mathsf{X} = q\,\mathsf{X}\mathsf{Y}\,, \tag{2}$$

then the following identities hold:

$$\langle \mathsf{X} \rangle_q \, \langle \mathsf{Y} \rangle_q = \langle \mathsf{X}+\mathsf{Y} \rangle_q \,, \tag{3}$$

$$\langle \mathsf{X} \rangle_q \, \langle \mathsf{XY} \rangle_q \, \langle \mathsf{Y} \rangle_q = \langle \mathsf{Y} \rangle_q \, \langle \mathsf{X} \rangle_q \,. \tag{4}$$

The first one is due to Schützenberger [9]. The second identity was found in [3] and is now commonly called the *pentagon identity*. It is also often called a *quantum dilogarithm identity* because it is closely related to the five–term dilogarithm identity [3, 11].

Now take three pairwise q–commuting indeterminates X, Y, and Z,

$$\mathsf{Y}\mathsf{X} = q\,\mathsf{X}\mathsf{Y}\,, \qquad \mathsf{X}\mathsf{Z} = q\,\mathsf{Z}\mathsf{X}\,, \qquad \mathsf{Z}\mathsf{Y} = q\,\mathsf{Y}\mathsf{Z}\,. \tag{5}$$

Following [6, 10], we utilize the pentagon relation twice and find that

$$\langle \mathsf{X} \rangle_q \, \langle \mathsf{Z} \rangle_q \, \langle \mathsf{XY} \rangle_q \, \langle \mathsf{Y} \rangle_q = \langle \mathsf{Z} \rangle_q \, \langle \mathsf{ZX} \rangle_q \, \langle \mathsf{X} \rangle_q \, \langle \mathsf{XY} \rangle_q \, \langle \mathsf{Y} \rangle_q = \langle \mathsf{Z} \rangle_q \, \langle \mathsf{ZX} \rangle_q \, \langle \mathsf{Y} \rangle_q \, \langle \mathsf{X} \rangle_q\,. \tag{6}$$

Note that XY and Z commute. Permuting $\langle \mathsf{XY} \rangle_q$ and $\langle \mathsf{Z} \rangle_q$ on the l.h.s, we bring identity (6) to the form $\mathsf{T} = \rho(\mathsf{T})$, where $\mathsf{T} = \langle \mathsf{X} \rangle_q \, \langle \mathsf{XY} \rangle_q \, \langle \mathsf{Z} \rangle_q \, \langle \mathsf{Y} \rangle_q$ and $\rho : \mathsf{X} \to \mathsf{Z}, \mathsf{Y} \to \mathsf{X}, \mathsf{Z} \to \mathsf{Y}$ is a third order automorphism of the associative algebra defined by presentation (5). Hence follows immediately the following triple identity

$$\mathsf{T} = \rho(\mathsf{T}) = \rho\left(\rho(\mathsf{T})\right) \,, \tag{7}$$

or, explicitly,

$$\langle \mathsf{X} \rangle_q \, \langle \mathsf{XY} \rangle_q \, \langle \mathsf{Z} \rangle_q \, \langle \mathsf{Y} \rangle_q = \langle \mathsf{Z} \rangle_q \, \langle \mathsf{ZX} \rangle_q \, \langle \mathsf{Y} \rangle_q \, \langle \mathsf{X} \rangle_q = \langle \mathsf{Y} \rangle_q \, \langle \mathsf{YZ} \rangle_q \, \langle \mathsf{X} \rangle_q \, \langle \mathsf{Z} \rangle_q\,. \tag{8}$$

In view of (7), we will say that (8) is a *cyclic* quantum dilogarithm identity.

The goal of this paper is to obtain a hierarchy of cyclic quantum dilogarithm identities in which (8) would be the first nontrivial member. For this purpose we will introduce an algebra $\mathcal{T}_N$ with generators assigned to the vertices of a certain quiver Q_N. In what follows, N stands for an integer number greater than one.

Definition 1. *The quiver Q_N is an oriented graph with vertices which are labelled by pairs of integer numbers (i, j) such that $1 \leq i < j \leq N$. The directed edges go from (i, j) to $(i, j+1)$, from (i, j) to $(i+1, j)$, and from $(i+1, j+1)$ to (i, j).*

Thus, Q_N has $\binom{N}{2}$ vertices. For instance, the quivers Q_3 and Q_4 are

$$
Q_3: \quad (1,2) \longrightarrow (1,3) \qquad\qquad (2,3)
$$

$$
Q_4: \quad
\begin{array}{c}
(1,2) \longrightarrow (1,3) \longrightarrow (1,4) \\
(2,3) \longrightarrow (2,4) \\
(3,4)
\end{array}
$$
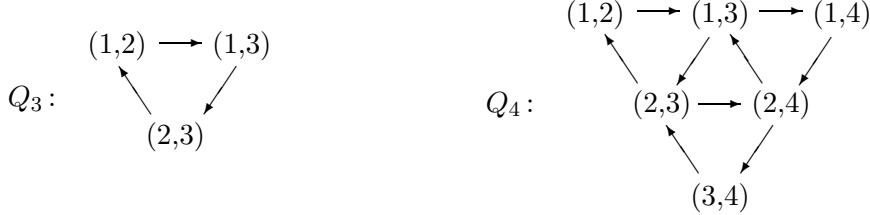
Given Q_N, one defines in the standard way its skew–symmetric incidence matrix B: $B_{(i,j),(i',j')} = -B_{(i',j'),(i,j)} = 1$ if there is a directed edge going from (i, j) to (i', j'), and $B_{(i,j),(i',j')} = 0$ if the vertices (i, j) and (i', j') are not connected.

Definition 2. *The algebra $\mathcal{T}_N$ associated with the quiver Q_N is a unital associative algebra over $\mathbb{R}$ with $\binom{N}{2}$ generators Z_{ij}, $1 \leq i < j \leq N$, and the following defining relations:*

$$
\mathsf{Z}_{ij}\,\mathsf{Z}_{i'j'} = q^{B_{(i,j),(i',j')}}\,\mathsf{Z}_{i'j'}\,\mathsf{Z}_{ij}\,. \tag{9}
$$

In particular, $\mathcal{T}_2$ is generated by a single generator. The algebra $\mathcal{T}_3$ (with generators Z_{12}, Z_{13}, Z_{23} renamed X, Z, Y) coincides with the algebra defined by presentation (5).

Remark 1. Let I be a subset of vertices of Q_N. Consider a linear homomorphism σ_I such that $\sigma_I(\mathsf{Z}_{ij}) = 0$ if $(i, j) \in I$ and $\sigma_I(\mathsf{Z}_{ij}) = \mathsf{Z}_{ij}$ otherwise. Since relations (9) are homogeneous, σ_I is an algebra homomorphism from $\mathcal{T}_N$ to its subalgebra. In particular, sending Z_{iN} to zero for all i, we reduce $\mathcal{T}_N$ to $\mathcal{T}_{N-1}$. Thus, we have a chain of subalgebra inclusions: $\mathcal{T}_2 \subset \mathcal{T}_3 \ldots \subset \mathcal{T}_{N-1} \subset \mathcal{T}_N$.

Remark 2. The center $\mathcal{Z}_N$ of the algebra $\mathcal{T}_N$ has dimension $N/2$ if N is even and $(N-1)/2$ if N is odd. (See Section 2.2 for more details.)

The quiver Q_N is mapped to itself by a clockwise rotation about its geometric center by $2\pi/3$. Hence, the algebra $\mathcal{T}_N$ admits a third order automorphism ρ,

$$
\rho\big(\mathsf{Z}_{ij}\big) = \mathsf{Z}_{j-i,N+1-i}\,. \tag{10}
$$

Furthermore, the quiver Q_N is mapped into a dual quiver (were all the arrows are reverted) by a reflection with respect to either of its three axes of symmetry passing through the corner vertices. Hence, the algebra $\mathcal{T}_N$ admits three involutive anti–automorphisms:

$$
\mu_1\big(\mathsf{Z}_{ij}\big) = \mathsf{Z}_{j-i,j} \qquad \mu_2\big(\mathsf{Z}_{ij}\big) = \mathsf{Z}_{N+1-j,N+1-i} \qquad \mu_3\big(\mathsf{Z}_{ij}\big) = \mathsf{Z}_{i,N+1+i-j}\,. \tag{11}
$$

Note that

$$
\mu_1 \circ \mu_2 = \mu_2 \circ \mu_3 = \mu_3 \circ \mu_1 = \rho\,. \tag{12}
$$

Let us adopt the following notations for products of non–commuting factors. Let $\prec$ be the lexicographic order relation of elements of a set $\Lambda \subset \mathbb{Z}^m$. Then $\overrightarrow{\prod\limits_{\lambda \in \Lambda}} f_\lambda$ and $\overleftarrow{\prod\limits_{\lambda \in \Lambda}} f_\lambda$ stand for ordered products, where f_λ is put to the right (respectively, to the left) of all $f_{\lambda'}$ such that $\lambda' \prec \lambda$. In particular, if $\Lambda = [1, k] \subset \mathbb{Z}$, then $\overrightarrow{\prod\limits_{\lambda \in \Lambda}} f_\lambda = f_1 \ldots f_k$ and $\overleftarrow{\prod\limits_{\lambda \in \Lambda}} f_\lambda = f_k \ldots f_1$.

Let $\Lambda_N \subset \mathbb{Z}^3$ be the following discrete tetrahedron containing $\binom{N+1}{3}$ points:

$$\Lambda_N = \left\{ \lambda = (a,b,c) \mid 1 \le a < b < c \le N+1 \right\}. \tag{13}$$

To each point $\lambda = (a,b,c) \in \Lambda_N$ we associate the following element of $\mathcal{T}_N$:

$$\mathbb{R}_\lambda = \left\langle \overrightarrow{\prod_{0 \le k \le c-b-1}} \mathsf{Z}_{a+k,b+k} \right\rangle_q. \tag{14}$$

Now we can define an analogue of the element T used in (7).

Definition 3. $\mathsf{T}_N \in \mathcal{T}_N$ is the following lexicographically ordered product:

$$\mathsf{T}_N = \overrightarrow{\prod_{\lambda \in \Lambda_N}} \mathbb{R}_\lambda. \tag{15}$$

In particular, we have $\mathsf{T}_2 = \mathbb{R}_{123} = \langle \mathsf{Z}_{12} \rangle_q$ and $\mathsf{T}_3 = \mathbb{R}_{123}\mathbb{R}_{124}\mathbb{R}_{134}\mathbb{R}_{234} = \langle \mathsf{Z}_{12} \rangle_q \langle \mathsf{Z}_{12}\mathsf{Z}_{23} \rangle_q \langle \mathsf{Z}_{13} \rangle_q \langle \mathsf{Z}_{23} \rangle_q$.

Remark 3. Some factors in the product (15) mutually commute, as, e.g., $\mathbb{R}_{124}$ and $\mathbb{R}_{134}$ in T_3. Therefore, T_N admits a number of equivalent expressions obtained by permutations of such factors. Some of these expressions are given in Lemma 2.

We will say that a quantum dilogarithm identity is of the type $n \sim m$ if it involves n quantum exponentials with monomial arguments on one side and m on the other. For instance, (4) and (8) are of the type $2 \sim 3$ and $4 \sim 4$, respectively. The main result of the present work is a family of cyclic quantum dilogarithm identities of the type $\binom{N+1}{3} \sim \binom{N+1}{3}$.

Theorem 1. *Identities*

$$\mathsf{T}_N = \mu_1\big(\mathsf{T}_N\big) = \mu_2\big(\mathsf{T}_N\big) = \mu_3\big(\mathsf{T}_N\big), \tag{16}$$

$$\mathsf{T}_N = \rho\big(\mathsf{T}_N\big) = \rho\big(\rho\big(\mathsf{T}_N\big)\big), \tag{17}$$

hold for any integer $N \ge 2$.

Remark 4. Identities (16) involve the anti–automorphism transformations μ_a corresponding to a reflection rather than a rotation symmetry of the quiver. Nevertheless, two of these identities are almost cyclic in the following sense. We will see below (cf. Lemma 8) that μ_2 applied to T_N acts almost as an identical transformation just permuting some commuting quantum exponentials. (As an example, apply to (8) an anti–automorphism which maps Z to itself and exchanges X and Y.) Along with (12) this implies that, again up to a permutation of commuting factors, μ_1 applied to T_N acts as ρ, and μ_3 applied to T_N acts as $\rho^{-1} = \rho \circ \rho$.

It is clear from the definition (1) of the quantum exponential that $\langle 0 \rangle_q = 1$. This, along with Remark 1, implies the following.

Proposition 1. *Let I be a subset of the set of vertices of Q_N. In the identity (17), replace with unity every quantum exponential that contains at least one Z_{ij} with $(i,j) \in I$. The result is a correct quantum dilogarithm identity.*

Note that the resulting reduced identity is not necessarily cyclic. For instance, sending Z to 0 in the $4 \sim 4$ identity (8), we obtain the pentagon identity (4).

Quantum dilogarithm identities associated with various quivers have close connections with (quantum) cluster algebras and Y–systems, see, e.g. [1, 4, 7, 8]. Our motivation for studying a specific family of such identities is an observation that they are *accessible* from the $4 \sim 4$ identities (8). For instance, let us demonstrate how to derive the equality $\mathsf{T}_4 = \rho(\mathsf{T}_4)$ without invoking the pentagon relation but using only the $4 \sim 4$ identities (8):

$$
\begin{aligned}
\mathsf{T}_4 &= \langle \mathsf{Z}_{12}\rangle_q \, \langle \mathsf{Z}_{12}\mathsf{Z}_{23}\rangle_q \, \langle \mathsf{Z}_{12}\mathsf{Z}_{23}\mathsf{Z}_{34}\rangle_q \, \langle \mathsf{Z}_{13}\rangle_q \, \langle \mathsf{Z}_{13}\mathsf{Z}_{24}\rangle_q \, \langle \mathsf{Z}_{14}\rangle_q \, \langle \mathsf{Z}_{23}\rangle_q \, \langle \mathsf{Z}_{23}\mathsf{Z}_{34}\rangle_q \, \langle \mathsf{Z}_{24}\rangle_q \, \langle \mathsf{Z}_{34}\rangle_q \\
&= \underline{\langle \mathsf{Z}_{12}\rangle_q \, \langle \mathsf{Z}_{12}\mathsf{Z}_{23}\rangle_q \, \langle \mathsf{Z}_{13}\rangle_q \, \langle \mathsf{Z}_{23}\rangle_q} \, \langle \mathsf{Z}_{12}\mathsf{Z}_{23}\mathsf{Z}_{34}\rangle_q \, \langle \mathsf{Z}_{13}\mathsf{Z}_{24}\rangle_q \, \langle \mathsf{Z}_{23}\mathsf{Z}_{34}\rangle_q \, \langle \mathsf{Z}_{14}\rangle_q \, \langle \mathsf{Z}_{24}\rangle_q \, \langle \mathsf{Z}_{34}\rangle_q \\
&= \langle \mathsf{Z}_{13}\rangle_q \, \langle \mathsf{Z}_{13}\mathsf{Z}_{12}\rangle_q \, \langle \mathsf{Z}_{23}\rangle_q \, \underline{\langle \mathsf{Z}_{12}\rangle_q \, \langle \mathsf{Z}_{12}\mathsf{Z}_{23}\mathsf{Z}_{34}\rangle_q \, \langle \mathsf{Z}_{13}\mathsf{Z}_{24}\rangle_q \, \langle \mathsf{Z}_{23}\mathsf{Z}_{34}\rangle_q} \, \langle \mathsf{Z}_{14}\rangle_q \, \langle \mathsf{Z}_{24}\rangle_q \, \langle \mathsf{Z}_{34}\rangle_q \\
&= \langle \mathsf{Z}_{13}\rangle_q \, \langle \mathsf{Z}_{23}\rangle_q \, \langle \mathsf{Z}_{13}\mathsf{Z}_{12}\rangle_q \, \langle \mathsf{Z}_{13}\mathsf{Z}_{24}\rangle_q \, \langle \mathsf{Z}_{13}\mathsf{Z}_{24}\mathsf{Z}_{12}\rangle_q \, \langle \mathsf{Z}_{23}\mathsf{Z}_{34}\rangle_q \, \langle \mathsf{Z}_{12}\rangle_q \, \langle \mathsf{Z}_{14}\rangle_q \, \langle \mathsf{Z}_{24}\rangle_q \, \langle \mathsf{Z}_{34}\rangle_q \\
&= \langle \mathsf{Z}_{13}\rangle_q \, \langle \mathsf{Z}_{23}\rangle_q \, \langle \mathsf{Z}_{13}\mathsf{Z}_{24}\rangle_q \, \underline{\langle \mathsf{Z}_{13}\mathsf{Z}_{12}\rangle_q \, \langle \mathsf{Z}_{13}\mathsf{Z}_{12}\mathsf{Z}_{24}\rangle_q \, \langle \mathsf{Z}_{14}\rangle_q \, \langle \mathsf{Z}_{24}\rangle_q} \, \langle \mathsf{Z}_{23}\mathsf{Z}_{34}\rangle_q \, \langle \mathsf{Z}_{34}\rangle_q \, \langle \mathsf{Z}_{12}\rangle_q \\
&= \langle \mathsf{Z}_{13}\rangle_q \, \langle \mathsf{Z}_{23}\rangle_q \, \langle \mathsf{Z}_{13}\mathsf{Z}_{24}\rangle_q \, \langle \mathsf{Z}_{14}\rangle_q \, \langle \mathsf{Z}_{14}\mathsf{Z}_{13}\mathsf{Z}_{12}\rangle_q \, \langle \mathsf{Z}_{24}\rangle_q \, \langle \mathsf{Z}_{13}\mathsf{Z}_{12}\rangle_q \, \langle \mathsf{Z}_{23}\mathsf{Z}_{34}\rangle_q \, \langle \mathsf{Z}_{34}\rangle_q \, \langle \mathsf{Z}_{12}\rangle_q \\
&= \langle \mathsf{Z}_{13}\rangle_q \, \langle \mathsf{Z}_{13}\mathsf{Z}_{24}\rangle_q \, \langle \mathsf{Z}_{14}\rangle_q \, \langle \mathsf{Z}_{14}\mathsf{Z}_{13}\mathsf{Z}_{12}\rangle_q \, \underline{\langle \mathsf{Z}_{23}\rangle_q \, \langle \mathsf{Z}_{23}\mathsf{Z}_{34}\rangle_q \, \langle \mathsf{Z}_{24}\rangle_q \, \langle \mathsf{Z}_{34}\rangle_q} \, \langle \mathsf{Z}_{13}\mathsf{Z}_{12}\rangle_q \, \langle \mathsf{Z}_{12}\rangle_q \\
&= \underline{\langle \mathsf{Z}_{13}\rangle_q \, \langle \mathsf{Z}_{13}\mathsf{Z}_{24}\rangle_q \, \langle \mathsf{Z}_{14}\rangle_q \, \langle \mathsf{Z}_{24}\rangle_q} \, \langle \mathsf{Z}_{14}\mathsf{Z}_{13}\mathsf{Z}_{12}\rangle_q \, \langle \mathsf{Z}_{24}\mathsf{Z}_{23}\rangle_q \, \langle \mathsf{Z}_{34}\rangle_q \, \langle \mathsf{Z}_{23}\rangle_q \, \langle \mathsf{Z}_{13}\mathsf{Z}_{12}\rangle_q \, \langle \mathsf{Z}_{12}\rangle_q \\
&= \langle \mathsf{Z}_{14}\rangle_q \, \langle \mathsf{Z}_{14}\mathsf{Z}_{13}\rangle_q \, \langle \mathsf{Z}_{24}\rangle_q \, \langle \mathsf{Z}_{13}\rangle_q \, \langle \mathsf{Z}_{14}\mathsf{Z}_{13}\mathsf{Z}_{12}\rangle_q \, \langle \mathsf{Z}_{24}\mathsf{Z}_{23}\rangle_q \, \langle \mathsf{Z}_{34}\rangle_q \, \langle \mathsf{Z}_{13}\mathsf{Z}_{12}\rangle_q \, \langle \mathsf{Z}_{23}\rangle_q \, \langle \mathsf{Z}_{12}\rangle_q \\
&= \langle \mathsf{Z}_{14}\rangle_q \, \langle \mathsf{Z}_{14}\mathsf{Z}_{13}\rangle_q \, \langle \mathsf{Z}_{14}\mathsf{Z}_{13}\mathsf{Z}_{12}\rangle_q \, \langle \mathsf{Z}_{24}\rangle_q \, \langle \mathsf{Z}_{24}\mathsf{Z}_{23}\rangle_q \, \langle \mathsf{Z}_{34}\rangle_q \, \langle \mathsf{Z}_{13}\rangle_q \, \langle \mathsf{Z}_{13}\mathsf{Z}_{12}\rangle_q \, \langle \mathsf{Z}_{23}\rangle_q \, \langle \mathsf{Z}_{12}\rangle_q \\
&= \rho(\mathsf{T}_4).
\end{aligned}
$$

The underlined terms were transformed by applying the $4 \sim 4$ identities (8). The remaining transformations changed only the order of commuting factors or the order of commuting generators in the arguments of quantum exponentials.

We will give a proof of the identities listed in Theorem 1 which makes it evident that they are accessible from the $4 \sim 4$ identities (8) for all N. The origin of this accessibility is that these identities stem from identities for certain words in a group whose generators satisfy the *tetrahedron equation*,

$$
R_{abc}\, R_{abd}\, R_{acd}\, R_{bcd} = R_{bcd}\, R_{acd}\, R_{abd}\, R_{abc}\,. \tag{18}
$$

The paper is organized as follows. Section 2 contains auxiliary statements which we need to combine together in order to prove Theorem 1. Namely, in Section 2.1 we consider families of groups $\mathcal{B}(n, N)$ whose generators satisfy the Yang–Baxter equation (for $n = 2$), the tetrahedron equation (for $n = 3$), or their higher analogues (for $n \geq 4$). The two key technical results here are an identity for certain words containing all the generators of the group $\mathcal{B}(n, N)$ and relation of these words to the element T_N. In Section 2.2 we describe the center of $\mathcal{T}_N$. In Section 2.3 we consider a local tensor space representation ϕ for $\mathcal{B}(3, N)$. In Section 2.4 we explain how evaluation of the above mentioned identity for words of $\mathcal{B}(3, N)$ in the representation ϕ yields ultimately the desired quantum dilogarithm identities for an arbitrary N. Appendix contains proofs of all statements given in Section 2.

2 Main technical ingredients

Below we assume that N and n are positive integers and $N \geq n$.

2.1 A group with n–simplex relations

Definition 4. $\mathcal{B}(n, N)$ *is a group with* $\binom{N}{n}$ *generators* $R_{a_1,\dots,a_n}$, *where* $1 \le a_1 < a_2 < \dots < a_n \le N$. *The group is defined by the following presentation:*

- *The generators commute,*

$$R_{a_1,\dots,a_n}\, R_{b_1,\dots,b_n} = R_{b_1,\dots,b_n}\, R_{a_1,\dots,a_n}\,, \tag{19}$$

  unless the set $\{a_1,\dots,a_n\} \bigcap \{b_1,\dots,b_n\}$ *contains exactly* $(n-1)$ *element.*

- *If* $N > n$, *the generators satisfy the following* $\binom{N}{n+1}$ *relations:*

$$\overrightarrow{\prod_{1 \le j \le n+1}} R_{a_1,\dots,\check{a}_j,\dots,a_{n+1}} = \overleftarrow{\prod_{1 \le j \le n+1}} R_{a_1,\dots,\check{a}_j,\dots,a_{n+1}}\,, \tag{20}$$

  where $\check{a}_j$ *is dropped.*

For $n = 1$, relations (20) imply commutativity, $R_a R_b = R_b R_a$, so that $\mathcal{B}(1, N)$ is an abelian group with N generators.

For $n = 2$, relations (20) have the form of the *Yang–Baxter equation,*

$$R_{ab}\, R_{ac}\, R_{bc} = R_{bc}\, R_{ac}\, R_{ab}\,, \qquad a < b < c\,, \tag{21}$$

and R_{ab} commutes with $R_{a'b'}$ if they have no common index.

For $n = 3$, relations (20) have the form of the *tetrahedron equation,*

$$R_{abc}\, R_{abd}\, R_{acd}\, R_{bcd} = R_{bcd}\, R_{acd}\, R_{abd}\, R_{abc}\,, \qquad a < b < c < d\,, \tag{22}$$

and R_{abc} commutes with $R_{a'b'c'}$ unless they have exactly two common indices.

Recall that, given a set $\Lambda \subset \mathbb{Z}^m$, we use the symbol $\overrightarrow{\prod_{\lambda \in \Lambda}} f_\lambda$ to denote the lexicographically ordered product of non–commuting factors.

Definition 5. *The word* $W(n, N) \in \mathcal{B}(n, N)$ *is the lexicographically ordered product of all generators of* $\mathcal{B}(n, N)$, *that is*

$$W(n, N) = \overrightarrow{\prod_{1 \le a_1 < a_2 \dots < a_n \le N}} R_{a_1,\dots,a_n}\,. \tag{23}$$

In particular, we have $W(n, n) = R_{1,\dots,n}$ for all n and $W(1, N) = R_1 \dots R_N$ for all N.

Let $\prec'$ be the colexicographic order relation of elements of a set $\Lambda \subset \mathbb{Z}^m$. That is, components of elements of Λ are compared starting from the right. For instance, $(a, b) \prec' (c, d)$ iff $b < d$ or $b = d$ and $a < c$. We will denote by $\overrightarrow{\prod}'_{\lambda \in \Lambda} f_\lambda$ and $\overleftarrow{\prod}'_{\lambda \in \Lambda} f_\lambda$ ordered products where f_λ is put to the right (respectively, to the left) of all $f_{\lambda'}$ such that $\lambda' \prec' \lambda$.

Lemma 1. *Define*

$$W'(n, N) = \overrightarrow{\prod_{1 \le a_1 < a_2 \dots < a_n \le N}}' R_{a_1,\dots,a_n}\,. \tag{24}$$

Then we have the equality

$$W(n, N) = W'(n, N)\,. \tag{25}$$

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For instance, (23) yields $W(2,4) = R_{12}R_{13}R_{14}R_{23}R_{24}R_{34}$ while (24) yields $W'(2,4) = R_{12}R_{13}R_{23}R_{14}R_{24}R_{34}$. These two words coincide since, by (19), R_{14} and R_{23} commute. It is also true in general that one needs to use only the commutativity relations (19) in order to change the order of factors in (24) to match that in (23).

Let η be an involutive anti–automorphism of $\mathcal{B}(n, N)$ such that

$$\eta\big(R_{a_1,\ldots,a_n}\big) = R_{a_1,\ldots,a_n} \tag{26}$$

for all the generators of $\mathcal{B}(n, N)$. Define

$$\bar{W}(n, N) = \eta\big(W(n, N)\big), \tag{27}$$

which is a word with the order of factors reverse to that of $W(n, N)$. Our first key technical statement is the following.

Theorem 2. *For all $N \geq n$, we have the equality*

$$W(n, N) = \bar{W}(n, N). \tag{28}$$

***Remark* 5.** A simple inspection of the proof given in the Appendix shows that the invertibility of R's is not really needed. Thus, Theorem 2 holds also if $\mathcal{B}(n, N)$ is a semigroup.

***Remark* 6.** The presented proof provides a constructive recursive procedure that transforms $W(n, N)$ into $\bar{W}(n, N)$. The transformation involves trivial moves based on (19) and moves $\mathcal{R}_{a_1,\ldots,a_{n+1}}$ which transform the l.h.s. of (20) into its r.h.s. For instance, $\mathcal{R}_{123}\big(W(2,4)\big) = R_{23}R_{13}R_{12}R_{14}R_{24}R_{34}$. Let $\mathcal{W}(n, N)$ and $\bar{\mathcal{W}}(n, N)$ stand for the compositions of such moves in which $\mathcal{R}$'s are ordered in the same way as R's are ordered in $W(n, N)$ and $\bar{W}(n, N)$. A simple inspection of the proof shows that the l.h.s. of (28) is transformed into its r.h.s. by $\mathcal{W}^*(n + 1, N)$, where star means that trivial moves are included when necessary. Moreover, the same transformation is achieved by $\bar{\mathcal{W}}^*(n + 1, N)$ if we start each reordering not from the left but from the right. Thus, the moves satisfy the identity $\mathcal{W}^*(n, N) = \bar{\mathcal{W}}^*(n, N)$ analogous to (28).

Eqs. (23) and (24) for $n = 2$ and $n = 3$ read

$$W(2, N) = \overset{\rightarrow}{\prod_{1 \leq a < b \leq N}} R_{ab}, \qquad W'(2, N) = \overset{\rightarrow}{\prod_{1 \leq a < b \leq N}}{}' R_{ab}, \tag{29}$$

$$W(3, N) = \overset{\rightarrow}{\prod_{1 \leq a < b < c \leq N}} R_{abc}, \qquad W'(3, N) = \overset{\rightarrow}{\prod_{1 \leq a < b < c \leq N}}{}' R_{abc}. \tag{30}$$

Recall that $\mathbb{R}_{abc}$ was defined in (14). We introduce also the following elements of $\mathcal{T}_N$:

$$\mathbb{R}_{ab} = \overset{\rightarrow}{\prod_{b+1 \leq c \leq N+1}} \mathbb{R}_{abc}. \tag{31}$$

Let us introduce the following homomorphisms from $\mathcal{B}(2, N)$ and $\mathcal{B}(3, N + 1)$ to $\mathcal{T}_N$ (it should be stressed that they are not algebra homomorphisms):

$$\sharp : R_{ab} \to \mathbb{R}_{ab}, \quad \star : R_{abc} \to \mathbb{R}_{abc}, \quad \star\star : R_{abc} \to \mathbb{R}_{a,c+a-b,c}. \tag{32}$$

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Lemma 2. *The element* T_N *defined in (15) can be obtained as follows:*

$$\mathsf{T}_N = \big(W(2,N)\big)^{\sharp} = \big(W'(2,N)\big)^{\sharp}, \tag{33}$$

$$\mathsf{T}_N = \big(W(3,N+1)\big)^{\star} = \big(W'(3,N+1)\big)^{\star}, \tag{34}$$

$$\mathsf{T}_N = \big(W(3,N+1)\big)^{\star\star} = \big(W'(3,N+1)\big)^{\star\star}. \tag{35}$$

This observation suggests that identities (28) can be recast in some quantum dilogarithm identities. Furthermore, Remark 6 indicates that one needs to use the $4 \sim 4$ relation (8) $\binom{N+1}{4}$ times in order to derive identity (17) for given N.

2.2 The center of $\mathcal{T}_N$

Let us assign to each vertex (i,j) of the quiver Q_N a non–negative integer weight α_{ij}. Let α denote the vector comprised of those weights taken in the lexicographic order. For instance, $\alpha = (\alpha_{12}, \alpha_{13}, \alpha_{23})$ if $N = 3$.

Monomials

$$\mathsf{M}(\alpha) = \overrightarrow{\prod_{1 \leq i < j \leq N}} \mathsf{Z}_{ij}^{\alpha_{ij}}, \qquad \alpha \in \mathbb{Z}_{\geq 0}^{N(N-1)/2} \tag{36}$$

constitute a basis of $\mathcal{T}_N$. We will say that α_{ij} are the weights of $\mathsf{M}(\alpha)$.

The algebra $\mathcal{T}_N$ has a nontrivial center $\mathcal{Z}_N$. In particular, for any N, if α^0 is a vector such that $\alpha_{ij}^0 = 1$, then $\mathsf{M}(\alpha^0)$ belongs to $\mathcal{Z}_N$.

Theorem 3. *Let $\chi(N)$ stand for $N/2$ if N is even and for $(N-1)/2$ if N is odd.*

- *If $\mathsf{M}(\alpha) \in \mathcal{Z}_N$, then its weights have the following symmetries*

$$\alpha_{ij} = \alpha_{j-i,N+1-i} = \alpha_{j-i,j} = \alpha_{N+1-j,N+1-i} = \alpha_{i,N+1+i-j}, \tag{37}$$

  and $\mathsf{M}(\alpha)$ is invariant under the action of ρ and μ_k defined in (10) and (11).

- *If $\mathsf{M}(\alpha), \mathsf{M}(\alpha') \in \mathcal{Z}_N$, and $\alpha_{1j} = \alpha'_{1j}$ for $j = 2, \ldots, \chi(N)+1$, then $\alpha = \alpha'$ and hence $\mathsf{M}(\alpha) = \mathsf{M}(\alpha')$.*

- *Given N and an arbitrary integer sequence $\beta_1, \ldots, \beta_{\chi(N)}$, there exist $m \in \mathbb{Z}_{\geq 0}$ and a vector $\alpha \in \mathbb{Z}_{\geq 0}^{N(N-1)/2}$ such that $\mathsf{M}(\alpha) \in \mathcal{Z}_N$ and $\alpha_{1j} = \beta_{j-1} + m$ for $j = 2, \ldots, \chi(N)+1$.*

In other words, each monomial central element $\mathsf{M}(\alpha) \in \mathcal{Z}_N$ is uniquely determined by its weights assigned to the vertices comprising a half of a boundary side of the quiver Q_N. It follows also that the dimension of $\mathcal{Z}_N$ is $\chi(N)$.

For instance, for $N = 3$ we have $\chi(3) = 1$ and $\mathsf{M}(\alpha) \in \mathcal{Z}_3$ iff $\alpha_{13} = \alpha_{23} = \alpha_{12}$. Thus, $\mathcal{Z}_3$ is generated by $\mathsf{Z}_{12}\mathsf{Z}_{13}\mathsf{Z}_{23}$. For $N = 4$ we have $\chi(4) = 2$ and $\mathsf{M}(\alpha) \in \mathcal{Z}_4$ iff $\alpha_{14} = \alpha_{34} = \alpha_{12}$ and $\alpha_{23} = \alpha_{24} = \alpha_{13}$. Thus, $\mathcal{Z}_4$ is generated by $\mathsf{Z}_{12}\mathsf{Z}_{14}\mathsf{Z}_{34}$ and $\mathsf{Z}_{13}\mathsf{Z}_{23}\mathsf{Z}_{24}$.

For $N = 5$ we have $\chi(5) = 2$ and $\mathsf{M}(\alpha) \in \mathcal{Z}_5$ iff $\alpha_{15} = \alpha_{45} = \alpha_{12}$, $\alpha_{14} = \alpha_{25} = \alpha_{35} = \alpha_{23} = \alpha_{34} = \alpha_{13}$, and $\alpha_{24} = 2\alpha_{13} - \alpha_{12}$. The last relation requires that $2\alpha_{13} \geq \alpha_{12}$.

Remark 7. The last part of Theorem 3 shows that the $\chi(N)$ weights that define a central monomial can be taken almost arbitrary (possibly, up to a total shift by an integer m). A direct inspection up to $N = 8$ suggests that in order to have $m = 0$ it suffices to take a non–decreasing integer sequence, $0 \leq \beta_1 \leq \ldots \leq \beta_{\chi(N)}$.

***Remark* 8.** The generators Z_{ij} of $\mathcal{T}_N$ can be constructed as exponential functions of canonically conjugate variables p_k, x_k such that $[x_k, p_{k'}] = \sqrt{-1}\,\delta_{kk'}$. Theorem 3 implies that the number of *degrees of freedom* of $\mathcal{T}_N$, i.e., the minimal number of such pairs (p_k, x_k) is $N(N-2)/4$ if N is even and $(N-1)^2/4$ if N is odd.

2.3 A local tensor space representation for tetrahedron equation

Let V be a vector space. For $N \geq 2$, we define $S_N = V^{\otimes N(N-1)/2}$. The tensor components of S_N will be labelled in the lexicographic order by pairs of integers (i, j), where $1 \leq i < j \leq N$. For instance, $S_3 = V_{12} \otimes V_{13} \otimes V_{23}$, where all V_{ij} are isomorphic to V.

For a given $\mathsf{R} \in \mathrm{End}\,(S_3)$, we denote by R_{abc} its canonical extension to a linear operator on S_N which acts non–trivially only on the tensor components V_{ab}, V_{ac}, and V_{bc}. E.g., $\mathsf{R}_{123} = \mathsf{R} \otimes id \otimes id \otimes id$ if $N = 4$.

Definition 6. *Let $\mathsf{R} \in \mathrm{End}\,(S_3)$ be invertible and let its action canonically extended to S_4 satisfy the tetrahedron equation,*

$$\mathsf{R}_{123}\,\mathsf{R}_{124}\,\mathsf{R}_{134}\,\mathsf{R}_{234} = \mathsf{R}_{234}\,\mathsf{R}_{134}\,\mathsf{R}_{124}\,\mathsf{R}_{123}\,. \tag{38}$$

A local tensor space representation of $\mathcal{B}(3, N)$ is a homomorphism $\mathcal{B}(3, N) \to \mathrm{End}\,(S_N)$ sending R_{abc} to R_{abc}.

Note that the commutativity relations (19) hold by construction. Indeed, if R_{abc} and $\mathsf{R}_{a'b'c'}$ have no common pair of indices then they act non–trivially on different tensor components of S_N and hence they commute.

***Remark* 9.** An equation formally identical to (38) was considered in [5]. There, however, it was treated as an operator equation on $V^{\otimes 4}$ with the identification $\mathsf{R}_{123} = \mathsf{R} \otimes id$ etc. We treat (38) as an equation on $V^{\otimes 6}$ which is the standard Zamolodchikov's tetrahedron equation [12] but with a non–standard (double index) labelling of the tensor components.

Let V be the vector space of formal series in x, x^{-1}. Then S_N is the vector space of formal series in x_{ij}, x_{ij}^{-1}, where $1 \leq i < j \leq N$.

Define operators $\mathsf{x}_{ij}, \mathsf{y}_{ij} \in \mathrm{End}\,(S_N)$, $1 \leq i < j \leq N$ such that

$$\begin{aligned}
\big(\mathsf{x}_{ij}\, f\big)(x_{12}, \ldots, x_{ij}, \ldots) &= x_{ij}\, f(x_{12}, \ldots, x_{ij}, \ldots)\,, \\
\big(\mathsf{y}_{ij}\, f\big)(x_{12}, \ldots, x_{ij}, \ldots) &= f(x_{12}, \ldots, qx_{ij}, \ldots)\,,
\end{aligned} \tag{39}$$

for any $f \in S_N$. These operators comprise $\binom{N}{2}$ q–commuting pairs,

$$\mathsf{x}_{ij}\,\mathsf{x}_{i'j'} = \mathsf{x}_{i'j'}\,\mathsf{x}_{ij}\,, \qquad \mathsf{y}_{ij}\,\mathsf{x}_{i'j'} = q^{\delta_{ii'}\delta_{jj'}}\,\mathsf{x}_{i'j'}\,\mathsf{y}_{ij}\,, \qquad \mathsf{y}_{ij}\,\mathsf{y}_{i'j'} = \mathsf{y}_{i'j'}\,\mathsf{y}_{ij}\,. \tag{40}$$

Consider $\mathsf{F} \in \mathrm{End}\,(S_3)$ whose action on monomials is given by

$$\mathsf{F} : x_{12}^k\, x_{13}^l\, x_{23}^m \to x_{12}^k\, x_{13}^{m+k}\, x_{23}^{l-k}\,, \tag{41}$$

or, equivalently, $(\mathsf{F}f)(x_{12}, x_{13}, x_{23}) = f(\frac{x_{12}x_{13}}{x_{23}}, x_{23}, x_{13})$. It is easy to check that

$$\mathsf{F}\,\mathsf{x}_{12} = \mathsf{x}_{12}\mathsf{x}_{13}\mathsf{x}_{23}^{-1}\mathsf{F}\,, \qquad \mathsf{F}\,\mathsf{x}_{13} = \mathsf{x}_{23}\,\mathsf{F}\,, \qquad \mathsf{F}\,\mathsf{x}_{23} = \mathsf{x}_{13}\,\mathsf{F}\,, \tag{42}$$

$$\mathsf{F}\,\mathsf{y}_{12} = \mathsf{y}_{12}\,\mathsf{F}\,, \qquad \mathsf{F}\,\mathsf{y}_{13} = \mathsf{y}_{12}\mathsf{y}_{23}\,\mathsf{F}\,, \qquad \mathsf{F}\,\mathsf{y}_{23} = \mathsf{y}_{12}^{-1}\mathsf{y}_{13}\,\mathsf{F}\,. \tag{43}$$

Clearly, F is invertible and $\mathsf{F}^2 = id$.

Let $\mathsf{R} \in \mathrm{End}\,(S_3)$ be the following operator

$$\mathsf{R} = \mathsf{F} \cdot \left\langle q^{1+\gamma} \mathsf{x}_{12} \mathsf{x}_{23}^{-1} \mathsf{y}_{12}^{\gamma} \mathsf{y}_{13}^{-\gamma} \mathsf{y}_{23}^{-1-\gamma} \right\rangle_q , \tag{44}$$

where $\gamma \in \mathbb{Z}$. A reader familiar with the Yang–Baxter equation will see in (44) an analogy with the standard ansatz for R–matrix, $\mathsf{R} = \mathsf{P} \cdot \check{\mathsf{R}}$, where P is the permutation, $\mathsf{P}^2 = id$.

Lemma 3. *Let F_{abc} and R_{abc} stand for the canonical extensions of F defined by (41) and R defined by (44) to operators on S_N. The homomorphisms $\theta, \phi_\gamma : \mathcal{B}(3, N) \to \mathrm{End}\,(S_N)$ such that $\theta(R_{abc}) = \mathsf{F}_{abc}$ and $\phi_\gamma(R_{abc}) = \mathsf{R}_{abc}$ are local tensor space representations of $\mathcal{B}(3, N)$.*

***Remark** 10.* This Lemma generalizes two previously known solutions of the tetrahedron equation. Namely, the case corresponding to $\gamma = 0$ was found in [10], and the case corresponding to $\gamma = -1$ was considered in [6]. Our proof follows closely that given in [6].

Lemma 4. *Let x_{ab}, y_{ab}, $1 \leq a < b \leq N+1$ act on S_{N+1} as defined in (39) and let $\mathsf{y}_{ab} \equiv 1$ if $a = b$. Then, for all $N \geq 2$, the linear homomorphism $\tau : \mathcal{T}_N \to \mathrm{End}\,(S_{N+1})$ such that*

$$\tau(Z_{ab}) = q \frac{\mathsf{x}_{ab}}{\mathsf{x}_{a+1,b+1}} \frac{\mathsf{y}_{a+1,b}}{\mathsf{y}_{a+1,b+1}} \qquad for \quad 1 \leq a < b \leq N , \tag{45}$$

is a faithful representation of $\mathcal{T}_N$.

2.4 Proof of Theorem 1

One the three symmetries of T_N presented in (16) can be established by reordering commuting factors. Namely, applying μ_2 to relations (34) and (35) and using Lemma 8 (see Appendix A.4), we obtain the following statement.

Lemma 5. *For all $N \geq 2$, we have*

$$\mu_2(\mathsf{T}_N) = \mathsf{T}_N . \tag{46}$$

Clearly, in order to establish an analogous equality involving μ_1 or μ_3 we will have to use the $4 \sim 4$ relation (8). To do it for μ_1, we establish a connection between the element T_N evaluated in the representation τ and words from $\mathcal{B}(3, N+1)$ evaluated in the representation ϕ_γ (we will take $\gamma = 0$ for simplicity).

Lemma 6. *For all $N \geq 2$, we have the equalities*

$$\phi_0\big(W'(3, N+1)\big) = \theta\big(W'(3, N+1)\big)\, \tau\big(\mathsf{T}_N\big) , \tag{47}$$

$$\phi_0\big(\bar{W}'(3, N+1)\big) = \theta\big(\bar{W}'(3, N+1)\big)\, \tau\big(\mu_1(\mathsf{T}_N)\big) . \tag{48}$$

Invoking Theorem 2, we infer that $\tau\big(\mathsf{T}_N\big) = \tau\big(\mu_1(\mathsf{T}_N)\big)$. Which, by virtue of Lemma 4, implies that

$$\mu_1(\mathsf{T}_N) = \mathsf{T}_N . \tag{49}$$

Now, combining (46) with (49) and using (12), we conclude that $\rho\big(\mathsf{T}_N\big) = (\mu_1 \circ \mu_2)\big(\mathsf{T}_N\big) = \mathsf{T}_N$. And finally, using (12) again, we obtain $\mu_3\big(\mathsf{T}_N\big) = (\mu_2 \circ \rho)\big(\mathsf{T}_N\big) = \mathsf{T}_N$.

Thus, we have obtained all relations given in Theorem 1.

A Appendix

A.1 Proofs of propositions of Section 2.1

Proof of Lemma 1. Let $\lfloor W \rfloor_{N+1} \in \mathcal{B}(n, N+1)$ denote the word which is obtained from a word $W \in \mathcal{B}(n-1, N)$ by the replacement $R_{a_1,\ldots,a_{n-1}} \to R_{a_1,\ldots,a_{n-1},N+1}$ applied to all factors. A key step in proving Lemma 1 is to observe that the following recursive relation holds

$$W(n, N) \lfloor W(n-1, N) \rfloor_{N+1} = W(n, N+1). \tag{50}$$

For instance,

$$W(2,3)\lfloor W(1,3) \rfloor_4 = R_{12}R_{13}R_{23}\lfloor R_1 R_2 R_3 \rfloor_4 = R_{12}R_{13}R_{14}R_{23}R_{24}R_{34} = W(2,4).$$

For $n = 2$, eq. (50) is almost trivial since $\lfloor W(1, N) \rfloor_{N+1} = R_{1,N+1}\ldots R_{N,N+1}$ and for each factor $R_{a,N+1}$, $a \neq N$ the rightmost factor in $W(2, N)$ with which it does not commute is $R_{a,N}$. Therefore, moving all factors from $\lfloor W(1, N) \rfloor_{N+1}$ to the left until they meet their non–commuting counterparts, we obtain the lexicographically ordered word $W(2, N+1)$. Note that only the last two factors in $\lfloor W(1, N) \rfloor_{N+1}$ have the same non–commuting counterpart, namely, $R_{N-1,N}$.

The proof for $n \geq 3$ is similar. Each factor $R_{a_1,\ldots,a_{n-1},N+1}$ in $\lfloor W(n-1, N) \rfloor_{N+1}$ belongs to a cluster, i.e. a product of consecutive lexicographically ordered factors. A cluster has length one if $a_{n-1} < N - 1$. If $a_{n-1} \geq N - 1$, then the length of a cluster is $k + 1$, where $k = \max\{m : a_{n-m} \geq N - m\}$. If $R_{a_1,\ldots,a_{n-k-1},N-k,\ldots,N-1,N+1}$ is the leftmost factor of a cluster of length $k + 1$, then the rightmost non–commuting counterpart in $W(n, N)$ for all factors of the cluster is $R_{a_1,\ldots,a_{n-k-1},N-k,\ldots,N-1,N}$ Therefore, moving each cluster from $\lfloor W(n-1, N) \rfloor_{N+1}$ to the left until it meets its counterpart, we achieve the lexicographic order of all factors, that is, we obtain the word $W(n, N+1)$.

The recursive relation (50) implies that

$$W(n, N) = \overset{\rightarrow}{\prod_{n \leq k \leq N}} \lfloor W(n-1, k-1) \rfloor_k, \tag{51}$$

which in turn leads to the expression (24) for $W'(n, N)$. $\qquad\square$

To prove Theorem 2 we will need also the following statement.

Lemma 7. *Let ω be an involutive automorphism of $\mathcal{B}(n, N)$ such that*

$$\omega\big(R_{a_1,\ldots,a_n}\big) = R_{N+1-a_n,\ldots,N+1-a_1}, \tag{52}$$

for all generators of $\mathcal{B}(n, N)$. Then for all $N \geq n$ the equalities

$$\omega\big(W(n, N)\big) = \eta\big(W'(n, N)\big), \qquad \omega\big(W'(n, N)\big) = \eta\big(W(n, N)\big) \tag{53}$$

hold in the strong sense, that is their r.h.s. coincide with their l.h.s. without a permutation of commuting factors.

Proof of Lemma 7. Denote $\bar{W}'(n, N) = \eta\big(W'(n, N)\big)$. Applying the anti–automorphism η to (24) and (50), we obtain

$$\bar{W}'(n, N) = \overset{\leftarrow}{\prod_{1 \leq b_1 < \ldots < b_n \leq N}}{}' R_{b_1,\ldots,b_n}, \tag{54}$$

$$\bar{W}'(n, N+1) = \lfloor \bar{W}'(n-1, N) \rfloor_{N+1} \bar{W}'(n, N). \tag{55}$$

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Applying to (54) the automorphism ω, and relabelling the indices by $a_k = N+1-b_{n+1-k}$, we recover the r.h.s of formula (23). Hence we infer that $\omega\big(\eta\big(W'(n,N)\big)\big) = W(n,N)$, which proves the first equality in (53). The second equality follows then immediately since ω and η mutually commute and both are involutive. □

Proof of Theorem 2. For $n = 1$, the statement of the Theorem is obvious since all R's commute. For $n = 2$, the statement can be proven by induction on N. The base, for $N = 3$, is simply the relation (20). Assume that the equality $W(2,N) = \bar{W}(2,N)$ has been already established for some $N \geq 3$. Then, taking into account the recursive structure of (50) and (55), we have to prove that

$$\Big\lfloor \overrightarrow{\prod_{1 \leq a \leq N}} R_a \Big\rfloor_{N+1} = \big(W(2,N)\big)^{-1} \Big\lfloor \overleftarrow{\prod_{1 \leq a \leq N}} R_a \Big\rfloor_{N+1} W(2,N) \,. \tag{56}$$

Observe that the Yang–Baxter equation (21) can be rewritten as the following "almost commutativity" relation for two R's,

$$\lfloor R_a R_b \rfloor_{N+1} = R_{ab}^{-1} \lfloor R_b R_a \rfloor_{N+1} R_{ab} \,. \tag{57}$$

Using this relation, we can move $R_{1,N+1}$ in the product $\overrightarrow{\prod} R_{a,N+1}$ to the right, then move $R_{2,N+1}$, and so on until we obtain the reverse ordered product $\overleftarrow{\prod} R_{a,N+1}$. Notice that all the extra "twisting" factors $R_{ab}^{\pm 1}$ arising in this process commute with any $R_{c,N+1}$, $c \neq a,b$. Therefore, at each step these twisting factors can be moved outside of the product $\lfloor \dots \rfloor_{N+1}$. It is easy to see that these factors combine into $\big(W(2,N)\big)^{\pm 1}$. Thus, we have established the inductive step (56) and hence the Theorem is proven for $n = 2$.

For $n = 3$, the Theorem can be proven along the same lines. The base, for $N = 4$, is the relation (20). Assuming that $W(3,N) = \bar{W}(3,N)$ has been already established for some $N \geq 4$ and taking into account (50) and (55), we have to prove the inductive step

$$\lfloor W(2,N) \rfloor_{N+1} = \big(W(3,N)\big)^{-1} \lfloor \bar{W}(2,N) \rfloor_{N+1} W(3,N) \,. \tag{58}$$

Observe that the tetrahedron equation (22) can be rewritten as the following "almost Yang–Baxter" relation for three R's,

$$\lfloor R_{ab} R_{ac} R_{bc} \rfloor_{N+1} = R_{abc}^{-1} \lfloor R_{bc} R_{ac} R_{ab} \rfloor_{N+1} R_{abc} \,, \tag{59}$$

where $a < b < c$. Notice that R_{abc} commutes with any factor in $\lfloor W(2,N) \rfloor_{N+1}$ except those on the l.h.s of (59). Therefore, the factors in $\lfloor W(2,N) \rfloor_{N+1}$ can be reordered exactly in the same way as the factors in $W(2,N)$ and the extra twisting factors $R_{abc}^{\pm 1}$ arising at each step can be moved outside of the product $\lfloor \dots \rfloor_{N+1}$. It is easy to see that these factors combine into $\big(W(3,N)\big)^{\pm 1}$. Since we already know that $W(2,N) = \bar{W}(2,N)$, it follows that the inductive step (58) holds and so the Theorem is proved for $n = 3$.

It is now clear that the proof continues by the double induction on n and N. For a given n, relation (20) can be rewritten as "almost" the relation for $n-1$. Using it, we can reorder the factors in $\lfloor W(n-1,N) \rfloor_{N+1}$ exactly in the same way as the factors in $W(n-1,N)$. At each step the extra twisting factors can be moved outside of the product $\lfloor \dots \rfloor_{N+1}$ due to relation (19). Therefore, the same line of arguments as for $n = 2, 3$ proves the Theorem for an arbitrary n. □

Proof of Lemma 2.

• The first equality in (33) is obvious from Definition 3 and eq. (31). In order to prove the second equality in (33), we consider the following elements of $\mathcal{T}_N$

$$\mathsf{T}_{N,b} = \prod_{1 \le a < k \le b}^{\rightarrow} \mathbb{R}_{ak}, \qquad 2 \le b \le N. \tag{60}$$

We observe that

$$\mathsf{T}_{N,b} = \mathsf{T}_{N,b-1} \prod_{1 \le a < b}^{\rightarrow} \mathbb{R}_{ab}. \tag{61}$$

Indeed, it follows from the definition (31) that the rightmost factor in the lexicographically ordered product $\mathsf{T}_{N,b}$ which does not commute with $\mathbb{R}_{ab}$ is $\mathbb{R}_{a,b-1}$. Repeating the argument used in the proof of formula (50) for $n = 2$, we conclude that (61) holds. And it is easy to see that (61) along with $\mathsf{T}_N = \mathsf{T}_{N,N}$ implies that

$$\mathsf{T}_N = \mathsf{T}_{N,N-1} \prod_{1 \le a < N}^{\rightarrow} \mathbb{R}_{aN} = \ldots = \prod_{1 \le a < b \le N}^{\rightarrow}{}' \mathbb{R}_{ab} = \left(W'(2,N)\right)^{\sharp}. \tag{62}$$

• The first equality in (34) is obvious from Definition 3. The second equality in (34) is equivalent to the statement that

$$\mathsf{T}_N = \mathsf{T}_{N-1} \mathsf{W}'(2,N), \tag{63}$$

where $\mathsf{W}'(2,N)$ is given by (29) with each R_{ab} replaced with $\mathbb{R}_{a,b,N+1}$. Using relations (50) and (62), we can rewrite the r.h.s. of (63) as follows

$$\tilde{\mathsf{W}}'(2, N-1) \mathsf{W}'(2, N-1) \left(\prod_{1 \le e < N}^{\rightarrow} \mathbb{R}_{eN} \right), \tag{64}$$

where $\tilde{\mathsf{W}}'(2,N)$ is given by (29) with each R_{ab} replaced with $\mathbb{R}'_{ab}$ which is an element of $\mathcal{T}_{N-1}$ given by (31). Definition (31) implies that $\mathbb{R}'_{ab}$ is the rightmost factor in $\tilde{\mathsf{W}}'(2, N-1)$ which does not commute with the factor $\mathbb{R}_{a,b,N+1}$ in $\mathsf{W}'(2, N-1)$. Therefore, moving all factors from $\mathsf{W}'(2, N-1)$ to the left until they meet their non–commuting counterparts and noticing that $\mathbb{R}'_{ab}\mathbb{R}_{a,b,N+1} = \mathbb{R}_{ab}$, we conclude that (64) coincides with the second expression in (62) and hence relation (63) holds.

• To prove the last part of the Lemma we consider $\mathsf{W}'(2, N)$ introduced in (63). One can check that all its factors containing $\mathsf{Z}_{N-1,N}$ can be moved to the right preserving their order. Making then the same procedure with factors containing $\mathsf{Z}_{N-2,N}$, $\mathsf{Z}_{N-3,N}$, etc., we conclude that $\mathsf{W}'(2, N)$ is given by the same expression in (29) where each R_{ab} is replaced with $\mathbb{R}_{a,N+1+a-b,N+1}$. With such a form of $\mathsf{W}'(2, N)$ formula (63) leads immediately to the second equality in (35).

Equivalence of the first and second expressions for T_N in (35) is verified as follows

$$\left(W(3,N)\right)^{\star\star} = \mu_2\left(\left(W'(3,N)\right)^{\star}\right) = \mu_2\left(\left(W(3,N)\right)^{\star}\right) = \left(W'(3,N)\right)^{\star\star}. \tag{65}$$

The middle equality is due to the second part of the Lemma. In the first and last equalities we used Lemma 8, which is proven below in Section A.4. $\qquad\square$

A.2 Proofs of propositions of Section 2.2

Proof of Theorem 3. Recall that $\alpha \in \mathbb{Z}_{\geq 0}^{N(N-1)/2}$ is a vector comprised of the weights α_{ij} assigned to the vertices of Q_N ordered lexicographically. By (36), each α determines a monomial $\mathsf{M}(\alpha) \in \mathcal{T}_N$. Relations (9) imply that $\mathsf{M}(\alpha)$ commutes with all generators of $\mathcal{T}_N$ iff $B\alpha = 0$. Thus, we have to study the kernel of the incidence matrix B.

• Let $\mathcal{U}$ be the symmetry transformation of the vertices of Q_N such that $\mathcal{U}(i,j) = (j-i, N+1-i)$, and U be the matrix of the corresponding orthogonal transformation of the basis in $\mathbb{Z}_{\geq 0}^{N(N-1)/2}$. Applying ρ to (9), we infer that B commutes with U. Taking into account that U is orthogonal and B is skew–symmetric, we conclude that if $\alpha \in \mathrm{Ker}B$, then α is an eigenvector of U. However, $U^3 = 1$ and so the only real eigenvalue of U is one. Thus, $\mathsf{M}(\alpha) \in \mathcal{Z}_N$ implies that $U\alpha = \alpha$. Hence

$$\alpha_{\mathcal{U}(i,j)} = \alpha_{ij}, \tag{66}$$

which is the first symmetry in (37). Applying ρ to a central monomial $\mathsf{M}(\alpha)$ and taking this symmetry of its weights into account, we infer that $\rho\big(\mathsf{M}(\alpha)\big) = q^{\varepsilon(\alpha)}\mathsf{M}(\alpha)$, where the multiplicative constant appears due to reordering of the generators. However, the property $\rho \circ \rho \circ \rho = id$ implies that $q^{\varepsilon(\alpha)} = 1$ and thus $\rho\big(\mathsf{M}(\alpha)\big) = \mathsf{M}(\alpha)$.

Let μ be any of the anti–homomorphisms in (11), $\mathcal{K}$ be the corresponding symmetry transformation of the vertices of Q_N (e.g. $\mathcal{K}(i,j) = (j-i,j)$ for μ_1), and K be the matrix of the corresponding orthogonal transformation of the basis in $\mathbb{Z}_{\geq 0}^{N(N-1)/2}$. Applying μ to (9), we infer that $KBK = -B$. Hence B^2 commutes with K. Taking into account that both these matrices are real symmetric and $K^2 = 1$, we conclude that if $\alpha \in \mathrm{Ker}B$, then $K\alpha = \pm\alpha$. But $\alpha_{12} = \alpha_{1N} = \alpha_{N-1,N}$ thanks to (66). Therefore, $K\alpha = \alpha$. Hence

$$\alpha_{\mathcal{K}(i,j)} = \alpha_{ij}, \tag{67}$$

which yields the remaining symmetries in (37). Applying μ to a central monomial $\mathsf{M}(\alpha)$ and taking these symmetries into account, we obtain that $\mu\big(\mathsf{M}(\alpha)\big) = q^{\varepsilon'(\alpha)}\mathsf{M}(\alpha)$. The property $\mu \circ \mu = id$ implies that $q^{\varepsilon'(\alpha)} = 1$ and thus $\mu\big(\mathsf{M}(\alpha)\big) = \mathsf{M}(\alpha)$.

• Consider monomials $\mathsf{M}(\alpha)$ and $\mathsf{M}(\alpha')$ such that $\alpha_{1j} = \alpha'_{1j}$ for $j = 2, \ldots, N$. Then, in view of (66), all components of vector $\alpha'' = \alpha - \alpha'$ corresponding to boundary vertices of Q_N vanish. Suppose that $\mathsf{M}(\alpha)$ and $\mathsf{M}(\alpha')$ are central. Then $\mathsf{M}(\alpha'')$ commutes with all generators of $\mathcal{T}_N$. In particular, the condition that $\mathsf{M}(\alpha'')$ commutes with all Z_{1j} is equivalent to a system of equations: $\alpha''_{2,j+1} = \alpha''_{2,j}$. However, we already have $\alpha''_{23} = 0$. Therefore, we conclude that $\alpha''_{2j} = 0$ for $j = 3, \ldots, N$. This implies, in view of (66), that all components of α'' corresponding to next to boundary vertices of Q_N also vanish. Continuing this consideration similarly for Z_{2j}, Z_{3j} etc., we conclude that $\alpha'' = 0$. Thus, $\mathsf{M}(\alpha') = \mathsf{M}(\alpha)$, i.e. two central monomials coincide iff they have coinciding weights at one boundary of Q_N. Thanks to the symmetry (67), the latter condition is equivalent to a weaker condition: $\alpha'_{1j} = \alpha_{1j}$ for $j = 2, \ldots, \chi(N) + 1$.

• Let us show that, given arbitrary integers $\beta_1, \ldots, \beta_{\chi(N)}$, there exists a unique integer vector $\tilde{\alpha}$ such that $\tilde{\alpha}_{1j} = \beta_{j-1}$ for $j = 2, \ldots, \chi(N)+1$ and $\mathsf{M}(\tilde{\alpha})$ given by (36) (where some $\tilde{\alpha}_{ij}$ can be negative) commutes with all generators Z_{ij}. Let ∂Q_N stand for the set of all boundary vertices of Q_N. Forgetting about the edges, we have $Q_N = \partial Q_N \cup \partial Q_{N-3} \cup \ldots$.

First, given the weights $\tilde{\alpha}_{1j}$ for $j = 2, \ldots, \chi(N) + 1$, we extend them to weights at other vertices of ∂Q_N by the symmetries (66) and (67). Now, the requirement that $\mathsf{M}(\tilde{\alpha})$ commutes with all generators assigned to ∂Q_N fixes uniquely all weights at ∂Q_{N-3}.

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Indeed, $\mathsf{M}(\tilde{\alpha})$ commutes with Z_{1j} if $\tilde{\alpha}_{2,j+1} - \tilde{\alpha}_{2j} = \tilde{\alpha}_{1,j+1} - \tilde{\alpha}_{1,j-1}$. Taken into account that $\tilde{\alpha}_{23} = \tilde{\alpha}_{13}$, this set of equations determines all $\tilde{\alpha}_{2j}$ uniquely. Moreover, the symmetry (67) for $\tilde{\alpha}_{1j}$ induces the same symmetry for $\tilde{\alpha}_{2j}$. Finally, $\tilde{\alpha}_{2j}$ can be extended to weights at other vertices of ∂Q_{N-3} by the symmetry (66).

Similarly, given weights at ∂Q_{k+3} and ∂Q_k, the requirement that $\mathsf{M}(\tilde{\alpha})$ commutes with all generators assigned to ∂Q_k fixes uniquely all weights at ∂Q_{k-3}. Indeed, $\mathsf{M}(\tilde{\alpha})$ commutes with all Z_{ij} for a given i if $\tilde{\alpha}_{i+1,j+1} - \tilde{\alpha}_{i+1,j} = \tilde{\alpha}_{i,j+1} - \tilde{\alpha}_{i,j-1} + \tilde{\alpha}_{i-1,j-1} - \tilde{\alpha}_{i-1,j}$. These equations are resolved uniquely since we know the r.h.s. and several first values of $\tilde{\alpha}_{i+1,j}$ thanks to the symmetry (66).

Thus, the vector $\tilde{\alpha}$ is recovered uniquely from its first $\chi(N)$ components. However, it can happen that some $\tilde{\alpha}_{ij}$ are negative. In this case we take another vector, $\alpha = \tilde{\alpha} + m\,\alpha^0$, where α^0 is a vector such that $\alpha_{ij}^0 = 1$, and m is sufficiently large positive number to ensure positivity of all α_{ij}. Then we have $\mathsf{M}(\alpha) = q^{\varepsilon} \cdot \mathsf{M}(\tilde{\alpha})\big(\mathsf{M}(\alpha^0)\big)^m$. It remains to observe that $\mathsf{M}(\alpha^0) \in \mathscr{Z}_N$ and hence $\mathsf{M}(\alpha) \in \mathscr{Z}_N$. $\qquad\square$

A.3 Proofs of propositions of Section 2.3

Proof of Lemma 3. The statement of the Lemma for F follows from that the action of $\mathsf{F}_{123}\,\mathsf{F}_{124}\,\mathsf{F}_{134}\,\mathsf{F}_{234}$ and $\mathsf{F}_{234}\,\mathsf{F}_{134}\,\mathsf{F}_{124}\,\mathsf{F}_{123}$ on the monomial $x_{12}^a x_{13}^b x_{23}^c x_{14}^d x_{24}^e x_{34}^f$, yields the same result.

Note that R given by (44) is invertible. If the argument of the quantum exponential in (44) is denoted by X, then we have $\mathsf{R}^{-1} = \langle\mathsf{X}\rangle_q^{-1}\,\mathsf{F}$, where $\langle x\rangle_q^{-1}$ is the following formal series

$$\langle x\rangle_q^{-1} = \sum_{n \geq 0} \frac{q^{n(n-1)/2}\,x^n}{(1-q)\ldots(1-q^n)}\,. \tag{68}$$

In order to check that R satisfies the tetrahedron equation, one has to substitute R_{abc} into (38), move all F's to the left, and cancel the products of F's on the both sides of the equation by invoking the first part of the Lemma. Then one is left with the equality

$$\langle\mathsf{X}\rangle_q \langle\mathsf{X}\mathsf{Y}\rangle_q \langle\mathsf{Z}\rangle_q \langle\mathsf{Y}\rangle_q = \langle\mathsf{Z}\rangle_q \langle\mathsf{Z}\mathsf{X}\rangle_q \langle\mathsf{Y}\rangle_q \langle\mathsf{X}\rangle_q\,, \tag{69}$$

where

$$\mathsf{X} = q^{1+\gamma}\mathsf{x}_{12}\mathsf{x}_{23}^{-1}\mathsf{y}_{12}^{\gamma}\mathsf{y}_{13}^{-\gamma}\mathsf{y}_{23}^{-1-\gamma}, \quad \mathsf{Y} = q^{1+\gamma}\mathsf{x}_{23}\mathsf{x}_{34}^{-1}\mathsf{y}_{23}^{\gamma}\mathsf{y}_{24}^{-\gamma}\mathsf{y}_{34}^{-1-\gamma},$$
$$\mathsf{Z} = q^{1+\gamma}\mathsf{x}_{13}\mathsf{x}_{24}^{-1}\mathsf{y}_{13}^{\gamma}\mathsf{y}_{14}^{-\gamma}\mathsf{y}_{23}^{1+\gamma}\mathsf{y}_{24}^{-1-\gamma}. \tag{70}$$

These operators satisfy relations (5). Therefore, comparing the first equality in (8) with (69), we conclude that equality (69) holds. $\qquad\square$

Proof of Lemma 4. Note that $\tau(\mathsf{Z}_{ab})$ does not commute with $\tau(\mathsf{Z}_{a'b'})$ only in the following six cases: 1) $a' = a$ and $b' = b \pm 1$; 2) $a' - a = \pm 1$ and $b' = b$ or $b' = b + a' - a$. It is easy to check that in these cases $\tau(\mathsf{Z}_{ab})$ and $\tau(\mathsf{Z}_{a'b'})$ satisfy the defining relations (9).

Clearly, the representation (45) is faithful for $\mathcal{T}_2$ since in this case the algebra has only one generator. The proof of faithfulness of this representation in other cases will use the inclusion $\mathcal{T}_{N-1} \subset \mathcal{T}_N$ (cf. Remark 1). Let N be the minimal positive number such that the representation τ is not faithful for $\mathcal{T}_N$. Then there exists a polynomial H in $\binom{N}{2}$ variables such that $\mathsf{H} \equiv H(\tau(\mathsf{Z}_{12}), \ldots, \tau(\mathsf{Z}_{N-1,N})) = 0$.

Without a loss of generality we can assume that H has the form

$$\mathsf{H} = \sum_{k_1,\ldots,k_{N-1}\geq 0} B_{k_1\ldots k_{N-1}}\,\tau\big(\mathsf{Z}_{1N}^{k_1}\ldots\mathsf{Z}_{N-1,N}^{k_{N-1}}\big)\,\mathsf{H}_{k_1\ldots k_{N-1}}\,, \tag{71}$$

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where $B_{k_1 \ldots k_{N-1}} \neq 0$ at least for one set $\{k_1, \ldots, k_{N-1}\}$ such that $k_1 + \ldots + k_{N-1} > 0$. $\mathsf{H}_{k_1 \ldots k_{N-1}}$ stand for polynomials in the generators of $\mathcal{T}_{N-1}$ evaluated in the representation (45).

Acting by H on $f \in S_N \subset S_{N+1}$, we obtain

$$\mathsf{H} \cdot f = \sum_{k_1 \ldots k_{N-1} \geq 0} q^{\varepsilon(k_1, \ldots, k_{N-1})} B_{k_1 \ldots k_{N-1}} \left(\prod_{a=1}^{N-1} \left(\frac{x_{aN}}{x_{a+1,N+1}} \right)^{k_a} \right) \left(\tilde{\mathsf{H}}_{k_1 \ldots k_{N-1}} \cdot f \right), \quad (72)$$

where $\tilde{\mathsf{H}}_{k_1 \ldots k_{N-1}} = \mathsf{Y}_N \mathsf{H}_{k_1 \ldots k_{N-1}}$ with $\mathsf{Y}_N = \mathsf{y}_{2N} \ldots \mathsf{y}_{N-1,N}$. Note that all terms in the sum in (72) are linearly independent monomials in $x_{2,N+1}^{-1}, \ldots, x_{N,N+1}^{-1}$. Hence, $\mathsf{H} = 0$ implies that $\tilde{\mathsf{H}}_{k_1 \ldots k_{N-1}} \cdot f = 0$. Since Y_N is invertible, we conclude that $\mathsf{H}_{k_1 \ldots k_{N-1}}$ annihilates an arbitrary f from S_N and, thus, $\mathsf{H}_{k_1 \ldots k_{N-1}} = 0$. But this contradicts the assumption that the representation τ is faithful for $\mathcal{T}_{N-1}$. $\qquad \square$

A.4 Proof of propositions of Section 2.4

Lemma 8. *For all $N \geq 3$, the following equalities*

$$\mu_2 \left((W(3, N))^\star \right) = (W'(3, N))^{\star\star}, \qquad \mu_2 \left((W'(3, N))^\star \right) = (W(3, N))^{\star\star}, \quad (73)$$

hold in the strong sense that is their r.h.s. coincide with their l.h.s. without a permutation of commuting factors.

Proof of Lemma 8. The first equality in (73) is checked with the help of Lemma 7 as follows:

$$\mu_2 \left((W(3, N))^\star \right) = \mu_2 \left(\overrightarrow{\prod_{1 \leq a < b < c \leq N}} (R_{abc})^\star \right) = \overrightarrow{\prod_{1 \leq a < b < c \leq N}}' \mu_2 \left((\omega(R_{abc}))^\star \right)$$

$$= \overrightarrow{\prod_{1 \leq a < b < c \leq N}}' \mu_2 \left(\mathbb{R}_{N+1-c, N+1-b, N+1-a} \right) = \overrightarrow{\prod_{1 \leq a < b < c \leq N}}' \mathbb{R}_{a, c+a-b, c} = (W'(3, N))^{\star\star}.$$

The second equality can be checked in a similar way. $\qquad \square$

Recall that the homomorphisms θ and ϕ_γ were introduced in Lemma 3. In this subsection we will use the following notations:

$$\mathsf{R}_{abc} = \phi_0(R_{abc}) = \mathsf{F}_{abc} \cdot \check{\mathsf{R}}_{abc}, \qquad \mathsf{F}_{abc} = \theta(R_{abc}), \qquad \check{\mathsf{R}}_{abc} = \left\langle q \mathsf{x}_{ab} \mathsf{x}_{bc}^{-1} \mathsf{y}_{bc}^{-1} \right\rangle_q. \quad (74)$$

We will need also the following homomorphisms φ, φ' from $\mathcal{B}(3, N)$ to $\mathrm{End}(S_N)$:

$$\varphi(R_{abc}) = \left\langle \frac{q \mathsf{x}_{ab}}{\mathsf{x}_{a+c-b,c}} \prod_{k=1}^{c-b} \frac{\mathsf{y}_{a+k,b+k-1}}{\mathsf{y}_{a+k,b+k}} \right\rangle_q, \quad (75)$$

$$\varphi'(R_{abc}) = \left\langle q \left(\prod_{k=1}^{c-b} \frac{\mathsf{x}_{b-a,b+k-1}}{\mathsf{x}_{b-a+1,b+k}} \right) \frac{\mathsf{y}_{b-a+1,b}}{\mathsf{y}_{b-a+1,c}} \right\rangle_q, \quad (76)$$

where $\mathsf{y}_{ab} \equiv 1$ if $a = b$.

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Recall that the word $W'(3, N)$ is given by (30), and that $\bar{W}'(3, N) = \eta\big(W'(3, N)\big)$ is the word with reversely ordered factors. Let us introduce

$$\mathsf{F}_N = \theta\big(W'(3, N)\big) = \theta\big(\bar{W}'(3, N)\big), \tag{77}$$

where the equality of the expressions is due to Theorem 2.

Lemma 9. *For all $N \geq 3$, we have the equalities*

$$\phi_0\big(W'(3, N)\big) = \mathsf{F}_N \cdot \varphi\big(W'(3, N)\big), \tag{78}$$

$$\phi_0\big(\bar{W}'(3, N)\big) = \mathsf{F}_N \cdot \varphi'\big(\bar{W}'(3, N)\big). \tag{79}$$

Proof of Lemma 9. Eq. (51) implies that

$$\phi_0\big(W'(3, N)\big) = \overrightarrow{\prod_{3 \leq c \leq N}} \phi_0\Big(\lfloor W'(2, c-1)\rfloor_c\Big). \tag{80}$$

Consider $\check{\mathsf{R}}_{abc}$ entering the $\phi_0\big(\lfloor \ldots \rfloor_c\big)$ part. In order to establish (78) it suffices to prove that:

i) $\check{\mathsf{R}}_{abc}$ transforms into $\varphi(\mathsf{R}_{abc})$ when all F's from $\phi_0\big(\lfloor \ldots \rfloor_c\big)$ are moved to the left of this part;

ii) $\varphi(\mathsf{R}_{abc})$ does not change when, for any $k > c$, all F's from $\phi_0\big(\lfloor W'(2, k-1)\rfloor_k\big)$ are moved through it.

   Proof of i). Consider $\check{\mathsf{R}}_{abc}$ which is not the rightmost one (otherwise, it already has the form $\varphi(R)$). All F's to the right of it have the form $\mathsf{F}_{a'b'c}$, $\{a', b'\} \neq \{a, b\}$. By (42), they do not change x_{ab} but act on the first index of x_{bc} as permutations $P_{a'b'}$. From the recursive structure of the word $W'(2, c-1) = \lfloor W(1, 1)\rfloor_2 \ldots \lfloor W(1, c-2)\rfloor_{c-1}$ it follows that x_{bc} has to be pulled through the following chain of permutations:

$$P_{a+1,b} \ldots P_{b-1,b} \; P_{1,b+1} \ldots P_{b,b+1} \ldots P_{1,c-1} \ldots P_{c-2,c-1}.$$

The part $P_{a+1,b} \ldots P_{b-1,b}$ here is present if $a < b - 1$. It transforms x_{bc} into $\mathsf{x}_{a+1,c}$. The remaining permutations are grouped into $(c - 1 - b)$ shift operators $U_d = P_{d,d+1} \ldots P_{12}$, $b \leq d \leq c-2$. Each U_d increases the first index of $\mathsf{x}_{a+1,c}$ by one, thus, transforming it into $\mathsf{x}_{a+c-b,c}$.

   Consider now y_{bc} entering the argument of $\check{\mathsf{R}}_{abc}$. It is first transformed by $\mathsf{F}_{a+1,b,c}$ into $\frac{\mathsf{y}_{a+1,c}}{\mathsf{y}_{a+1,b}}$. Note that, by (43), all F's entering $\phi_0\big(\lfloor W'(2, c-1)\rfloor_c\big)$ act non–trivially only on y's whose second index is c. From the recursive structure of the word $W'(2, c-1)$ it follows that $\mathsf{y}_{a+1,c}$ is transformed by consecutive action of pairs $\mathsf{F}_{a+k,b+k,c}\mathsf{F}_{a+1+k,b+k,c}$, $1 \leq k \leq c - b - 1$. Each such pair transforms $\mathsf{y}_{a+k,c}$ into $\frac{\mathsf{y}_{a+k,b+k}}{\mathsf{y}_{a+1+k,b+k}}\mathsf{y}_{a+1+k,c}$, which yields the product of y's in (75).

   Proof of ii). This part is trivial for y's since, by (43), y_{ab} commutes with any $\mathsf{F}_{a'b'k}$ if $k \neq a, b$. For x's we have to consider the transformations of $\frac{\mathsf{x}_{ab}}{\mathsf{x}_{a'c}}$, where $a' \equiv a + c - b$. Note that $a < a' < c$. Therefore, $\frac{\mathsf{x}_{ab}}{\mathsf{x}_{a'c}}$ is first transformed by F_{abk} into $\frac{\mathsf{x}_{ab}\mathsf{x}_{ak}}{\mathsf{x}_{a'c}\mathsf{x}_{bk}}$. Then this expression is pulled through all F's between F_{abk} and $\mathsf{F}_{a'ck}$. They act only on the first indices of x_{ak} and x_{bk} as the following chain of permutations:

$$P_{a+1,b} \ldots P_{b-1,b} \; P_{1,b+1} \ldots P_{b,b+1} \ldots P_{1,c} \ldots P_{a'-1,c}.$$

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The part $P_{a+1,b} \ldots P_{b-1,b}$ transforms $\frac{\mathsf{x}_{ak}}{\mathsf{x}_{bk}}$ into $\frac{\mathsf{x}_{ak}}{\mathsf{x}_{a+1,k}}$. Then each of the shift operators U_d, $b \leq d \leq c-2$ increases the first indices of x_{ak} and $\mathsf{x}_{a+1,k}$ by one, which yields $\frac{\mathsf{x}_{a'-1,k}}{\mathsf{x}_{a'k}}$. The last part, $P_{1,c} \ldots P_{a'-1,c}$, transforms $\frac{\mathsf{x}_{a'-1,k}}{\mathsf{x}_{a'k}}$ into $\frac{\mathsf{x}_{ck}}{\mathsf{x}_{a'k}}$. Finally, $\frac{\mathsf{x}_{ab}\mathsf{x}_{ck}}{\mathsf{x}_{a'c}\mathsf{x}_{a'k}}$ is transformed by $\mathsf{F}_{a'ck}$ into $\frac{\mathsf{x}_{ab}}{\mathsf{x}_{a'c}}$ which then is not affected by the remaining F's.

Thus, we have verified **i)** and **ii)** and therefore proved relation (78).

In order to establish Eq. (79) we will prove first that

$$\phi_0\big(\bar{W}'(3,N)\big) = \varphi''\big(\bar{W}'(3,N)\big) \cdot \mathsf{F}_N\,, \tag{81}$$

where φ'' is a homomorphism from $\mathcal{B}(3,N)$ to $\mathrm{End}\big(S_N\big)$,

$$\varphi''(R_{abc}) = \left\langle \frac{q\mathsf{x}_{ab}}{\mathsf{x}_{a+c-b,c}} \prod_{k=0}^{c-b-1} \frac{\mathsf{y}_{a+k,b+k}}{\mathsf{y}_{a+k,b+k+1}} \right\rangle_q . \tag{82}$$

Indeed, we have

$$\mathsf{R}_{abc} = \phi_0(R_{abc}) = \hat{\mathsf{R}}_{abc} \cdot \mathsf{F}_{abc}\,, \qquad \hat{\mathsf{R}}_{abc} = \big\langle q\mathsf{x}_{ab}\mathsf{x}_{bc}^{-1}\mathsf{y}_{ab}\mathsf{y}_{ac}^{-1}\big\rangle_q . \tag{83}$$

Note that the x's arguments of $\hat{\mathsf{R}}_{abc}$ and $\check{\mathsf{R}}_{abc}$ are the same. Therefore, moving all F's in $\phi_0\big(\bar{W}'(3,N)\big)$ to the right results in the same transformation of these arguments as moving all F's in $\phi_0\big(W'(3,N)\big)$ to the left.

What the y's arguments of $\hat{\mathsf{R}}_{abc}$ are concerned, they are transformed only by those F's that enter the $\phi_0\big(\lfloor\bar{W}'(2,c-1)\rfloor_c\big)$ part. Specifically, y_{ab} does not change whereas y_{ac} is transformed by consecutive action of pairs $\mathsf{F}_{a+k,b+1+k,c}\mathsf{F}_{a+1+k,b+1+k,c}$, $0 \leq k \leq c-b-1$, which yields the product of y's in (82).

It remains to pull F_N in (81) to the left. Note that $\mathsf{F}_N^2 = 1$. Using (42) and (43), one can verify that

$$\mathsf{F}_N\,\mathsf{x}_{ab}\,\mathsf{F}_N = \mathsf{x}_{b-a,b} \prod_{k=1}^{N-b} \frac{\mathsf{x}_{b-a,b+k}}{\mathsf{x}_{b-a+1,b+k}}\,, \qquad \mathsf{F}_N\,\mathsf{y}_{ab}\,\mathsf{F}_N = \prod_{k=1}^{b-a} \frac{\mathsf{y}_{k,a+k}}{\mathsf{y}_{k,a+k-1}}\,. \tag{84}$$

Whence

$$\frac{\mathsf{x}_{ab}}{\mathsf{x}_{a+1,b+1}}\,\mathsf{F}_N = \mathsf{F}_N\,\frac{\mathsf{x}_{b-a,b}}{\mathsf{x}_{b-a+1,b+1}}\,, \qquad \frac{\mathsf{y}_{ab}}{\mathsf{y}_{a,b+1}}\,\mathsf{F}_N = \mathsf{F}_N\,\frac{\mathsf{y}_{b-a+1,b}}{\mathsf{y}_{b-a+1,b+1}}. \tag{85}$$

Applying these relations to (82), we obtain (76). $\qquad\qquad\square$

Proof of Lemma 6. Using (14), (32), and (45), we find

$$\tau\big(\mathbb{R}_{abc}\big) = \varphi\big(R_{abc}\big)\,, \qquad \tau\big(\mu_1\big(\mathbb{R}_{abc}\big)\big) = \varphi'\big(R_{abc}\big). \tag{86}$$

Therefore, invoking Lemma 2, we conclude that

$$\tau\big(\mathsf{T}_N\big) = \tau\big(\big(W'(3,N+1)\big)^\star\big) = \varphi\big(W'(3,N+1)\big)\,, \tag{87}$$

$$\tau\big(\mu_1\big(\mathsf{T}_N\big)\big) = \tau\big(\mu_1\big(\big(W'(3,N+1)\big)^\star\big)\big) = \varphi'\big(\bar{W}'(3,N+1)\big). \tag{88}$$

These relations along with Lemma 9 yield the statement of Lemma 6. $\qquad\square$

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